The interval (discrete) Fourier transform

Marco De Angelis

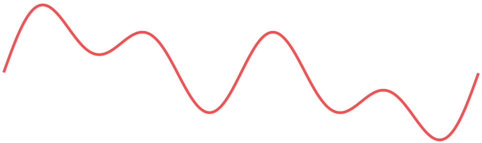
Institute for Risk and Uncertainty, University of Liverpool

May 17, 2021

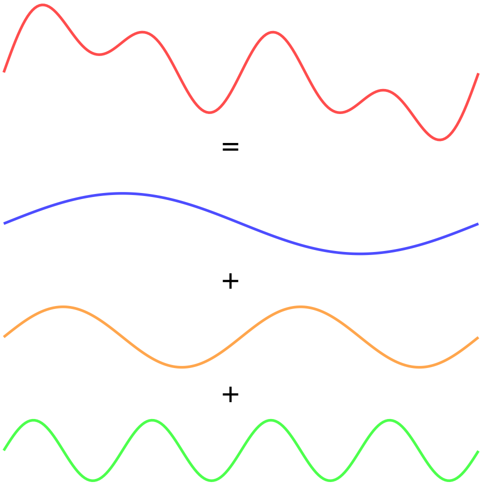
UNIVERSITY OF LIVERPOOL | Institute for Risk and Uncertainty

Outline

Motivations

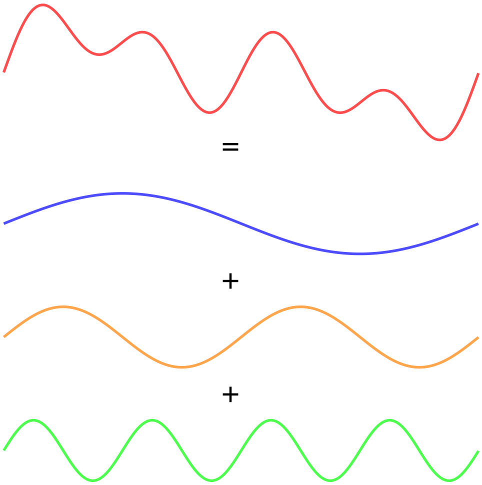


Motivations



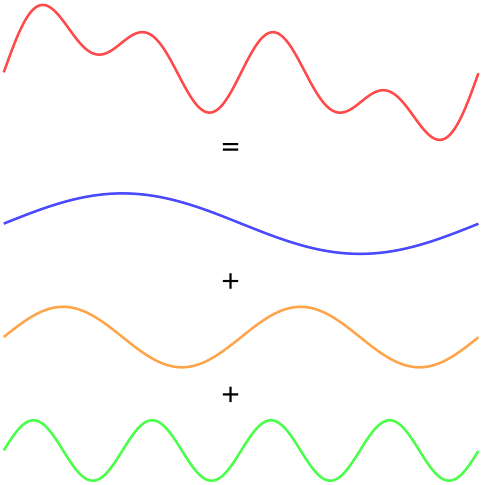
Motivations

- Imprecise/poor sensor measurements



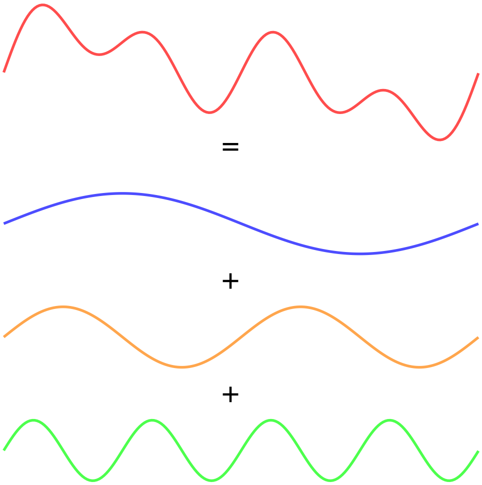
Motivations

- Imprecise/poor sensor measurements
- Missing data



Motivations

- Imprecise/poor sensor measurements
- Missing data
- Relaxed assumptions



Notation

- $\mathcal{F}$: Fourier transform
- $\mathcal{F}_h$: Fourier transform for a given harmonic h
- f: a function
- $[f]$: Natural extension
- $\hat{f}$: United extension

In this presentation we use capital letters to denote interval variables.

The discrete Fourier transform

$$\mathcal{F}_h : \mathbb{R}^n \to \mathbb{C}, \quad \forall h \in \mathbb{Z}_+$$

$$\mathcal{F}_h(x) := \sum_{k=0}^{n-1} x_k \; e^{-i\frac{2\pi}{n}hk}$$

The discrete Fourier transform

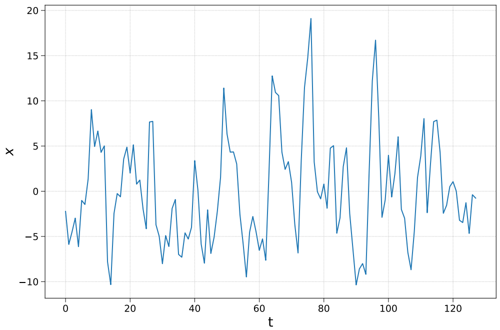
$$\mathcal{F}_h : \mathbb{R}^n \to \mathbb{C}, \quad \forall h \in \mathbb{Z}_+$$

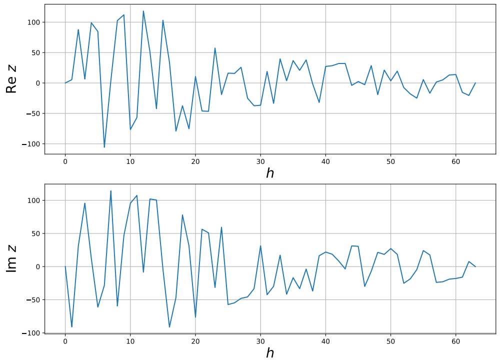
$$\mathcal{F}_h(x) := \sum_{k=0}^{n-1} x_k \; e^{-i \frac{2\pi}{n} hk}$$

or

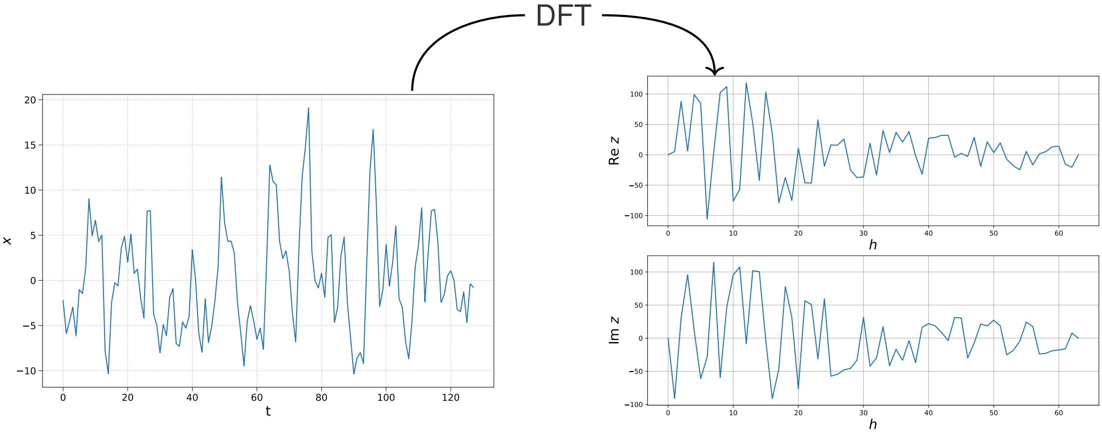
$$\mathcal{F}_h(x) := \sum_{k=0}^{n-1} x_k \; \left(\cos \frac{2\pi}{n} hk - i \sin \frac{2\pi}{n} hk \right)$$

The discrete Fourier transform

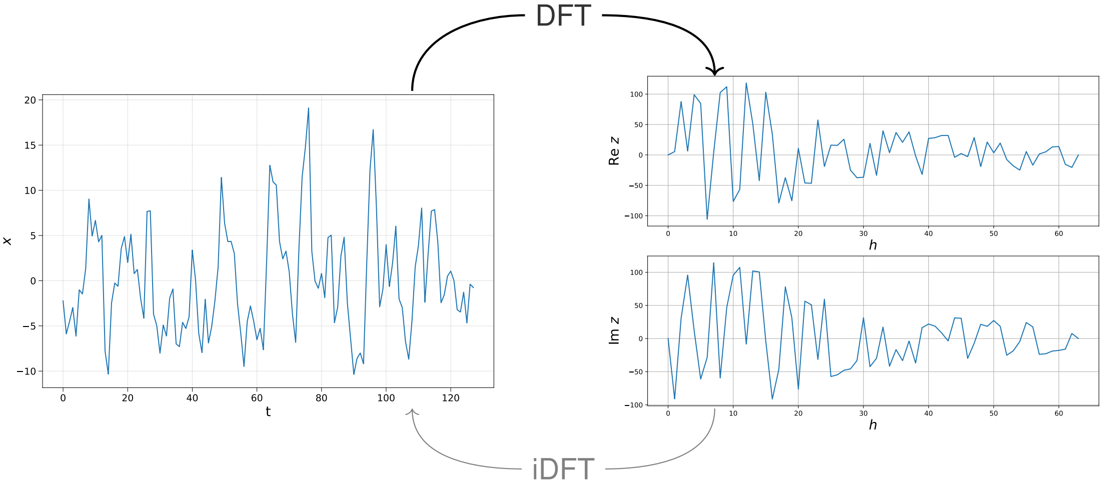




The discrete Fourier transform

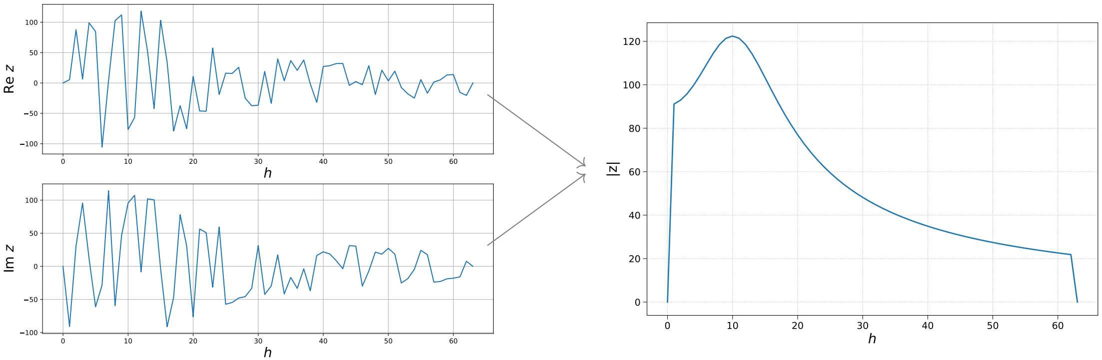


The discrete Fourier transform



Amplitude of the Fourier transform

$$\left| \sum_{k=0}^{n-1} x_k \; e^{-i \frac{2\pi}{n} hk} \right|$$



United extension $\hat{\mathcal{F}}$

$$\hat{\mathcal{F}} : S(A) \to S(B)$$

where, $S(A)$, $S(B)$ is the family of subset of A, B.

$$\hat{\mathcal{F}}(X) = \bigcup_{X \in S(A)} \{\mathcal{F}(x) \mid x \in X\}$$

United extensions are **inclusion monotonic**

Interval extension $[\mathcal{F}]$

$$\mathcal{F} : \mathbb{R}^n \to \mathbb{C}$$

$$[\mathcal{F}](\{x\}) = \mathcal{F}(x), \quad \forall x \in \mathbb{R}^n$$

where, $\{x\}$ is a degenerate interval.

Fundamental theorems of IA

Theorem 1 (United extension)

*If $[f]$ is an **inclusion monotonic** extension of f then,*

$$\hat{\mathcal{F}}(x) \subseteq [\mathcal{F}](x). \tag{1}$$

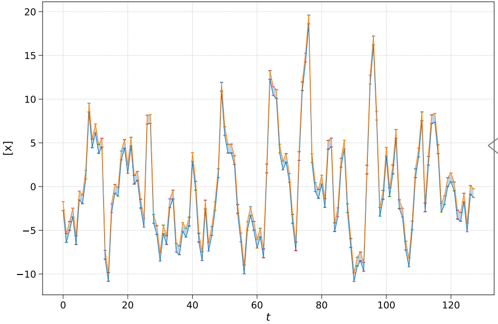
Theorem 2 (Natural extension)

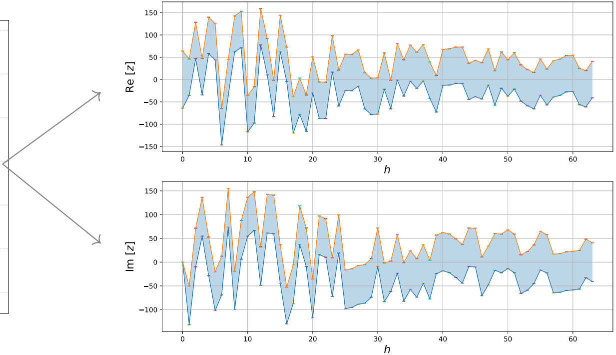
Any real function whose entries have been replaced with intervals is inclusion monotonic.

$$f(x) = x - x^2, \qquad [f](X) = X - X^2. \tag{2}$$

Interval discrete Fourier transform

$$\sum_{k=0}^{n-1} X_k \, e^{-i\frac{2\pi}{n}hk}$$





Computing $[\mathcal{F}]$

$$[\mathcal{F}_h](x) = \sum_{k=0}^{n-1} X_k \; e^{-i\frac{2\pi}{n}hk}$$

$$X = [X_1, X_2] = \{x \in \mathbb{R}^n | X_1 \leq x \leq X_2\}$$

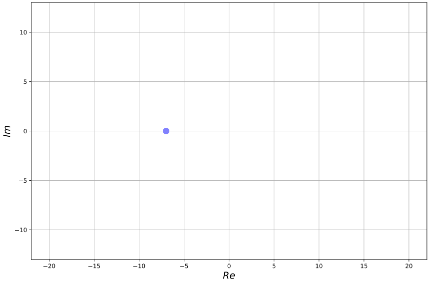
$$[\mathcal{F}] = \left[\inf_{X_1 \leq x \leq X_2} \mathcal{F}(x), \quad \sup_{X_1 \leq x \leq X_2} \mathcal{F}(x) \right]$$

For the united extension extra computational power is needed.

Example, $f : \mathbb{R}^6 \to \mathbb{C}, \qquad f(x) = b\, x^T$

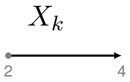
$$b = \begin{pmatrix} 1+i2 & -2+i2 & 3+i2 & \ldots \\ 4-i2 & -5-i2 & 6+i2 & \end{pmatrix}$$

$$X^T = \begin{pmatrix} [-1,1] \\ [-1,1] \\ [-1,1] \\ [-1,1] \\ [-1,1] \\ [-1,1] \end{pmatrix}$$

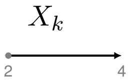


$$\sum_{k=0}^{n-1} X_k \left(\cos \frac{2\pi}{n} hk - i \sin \frac{2\pi}{n} hk \right)$$

$$\sum_{k=0}^{n-1} X_k \left(\cos \frac{2\pi}{n} hk - i \sin \frac{2\pi}{n} hk \right)$$



$$\sum_{k=0}^{n-1} X_k \left(\underset{r_k}{\boxed{\cos \frac{2\pi}{n} hk}} - i \underset{i_k}{\boxed{\sin \frac{2\pi}{n} hk}} \right)$$



$$\sum_{k=0}^{n-1} X_k \left(\underbrace{\boxed{\cos\frac{2\pi}{n}hk}}_{r_k} - i\,\underbrace{\boxed{\sin\frac{2\pi}{n}hk}}_{i_k}\right)$$

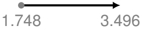
X_k

$r_k = 0.874$

2 4

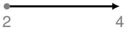
$i_k = 1.724$

$$\sum_{k=0}^{n-1} X_k \left(\underbrace{\boxed{\cos \frac{2\pi}{n} hk}}_{r_k} - i \underbrace{\boxed{\sin \frac{2\pi}{n} hk}}_{i_k} \right)$$



X_k

$r_k = 0.874$



$i_k = 1.724$

$$\sum_{k=0}^{n-1} X_k \left(\underbrace{\boxed{\cos\frac{2\pi}{n}hk}}_{r_k} - i\underbrace{\boxed{\sin\frac{2\pi}{n}hk}}_{i_k}\right)$$

1.748 3.496

X_k

$r_k = 0.874$

2 4

$i_k = 1.724$

-6.896 -3.448

$$\sum_{k=0}^{n-1} X_k \left(\underbrace{\boxed{\cos\frac{2\pi}{n}hk}}_{r_k} - i\,\underbrace{\boxed{\sin\frac{2\pi}{n}hk}}_{i_k}\right)$$

X_k

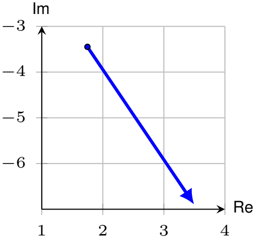
2 4

$r_k = 0.874$

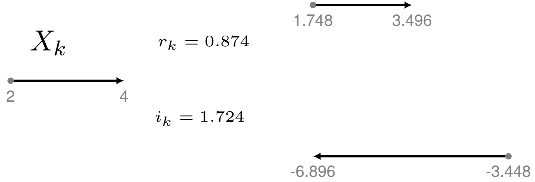
1.748 3.496

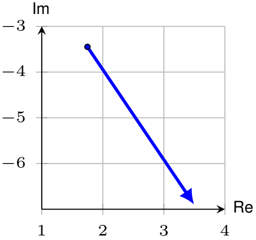
$i_k = 1.724$

-6.896 -3.448



$$\sum_{k=0}^{n-1} X_k \left(\underbrace{\boxed{\cos \frac{2\pi}{n} hk}}_{r_k} - i \underbrace{\boxed{\sin \frac{2\pi}{n} hk}}_{i_k} \right)$$





$$\sum_{k=0}^{n-1} X_k \cdot r_k + i\, X_k \cdot i_k, \quad r_k, i_k \in \mathbb{R}$$

$$\sum_{k=0}^{n-1} X_k \cdot r_k + i\, X_k \cdot i_k, \quad r_k, i_k \in \mathbb{R}$$

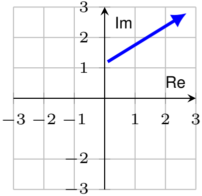
$$X_0 \cdot r_0 \quad + \quad X_1 \cdot r_1 \quad + \quad X_2 \cdot r_2 \quad + \quad ...$$

$$\sum_{k=0}^{n-1} X_k \cdot r_k + {\color{red}i}\, X_k \cdot i_k, \quad r_k, i_k \in \mathbb{R}$$

$$\begin{array}{lccccccc} & X_0 \cdot r_0 & + & X_1 \cdot r_1 & + & X_2 \cdot r_2 & + & \ldots \\ {\color{red}i} & X_0 \cdot i_0 & + & X_1 \cdot i_1 & + & X_2 \cdot i_2 & + & \ldots \end{array}$$

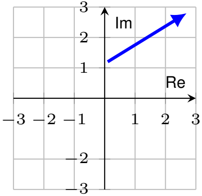
$$\sum_{k=0}^{n-1} X_k \cdot r_k + i\, X_k \cdot i_k, \quad r_k, i_k \in \mathbb{R}$$

$$\begin{array}{ccccccccc} & \boxed{\begin{array}{c} X_0 \cdot r_0 \\ X_0 \cdot i_0 \end{array}} & + & X_1 \cdot r_1 & + & X_2 \cdot r_2 & + & \ldots \\ i & & + & X_1 \cdot i_1 & + & X_2 \cdot i_2 & + & \ldots \end{array}$$



$$\sum_{k=0}^{n-1} X_k \cdot r_k + i\, X_k \cdot i_k, \quad r_k, i_k \in \mathbb{R}$$

$$\begin{array}{cccccccc} & \boxed{\begin{array}{c} X_0 \cdot r_0 \\ X_0 \cdot i_0 \end{array}} & + & X_1 \cdot r_1 & + & X_2 \cdot r_2 & + & \ldots \\ i & & + & X_1 \cdot i_1 & + & X_2 \cdot i_2 & + & \ldots \end{array}$$

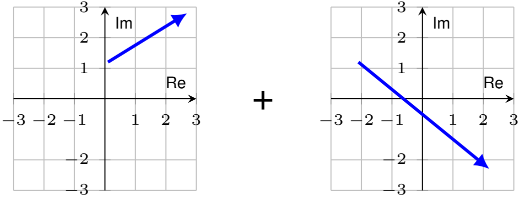


$+$

$$\sum_{k=0}^{n-1} X_k \cdot r_k + {\color{red} i}\, X_k \cdot i_k, \qquad r_k, i_k \in \mathbb{R}$$

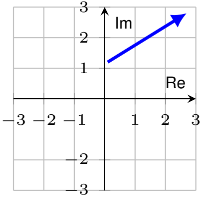
$$\begin{array}{cccccccc} & \boxed{\begin{array}{c} X_0 \cdot r_0 \\ X_0 \cdot i_0 \end{array}} & + & \boxed{\begin{array}{c} X_1 \cdot r_1 \\ X_1 \cdot i_1 \end{array}} & + & \begin{array}{c} X_2 \cdot r_2 \\ X_2 \cdot i_2 \end{array} & + & \begin{array}{c} \ldots \\ \ldots \end{array} \\ \end{array}$$

$${\color{red} i}$$ (applies to the second row: $X_0 \cdot i_0 + X_1 \cdot i_1 + X_2 \cdot i_2 + \ldots$)

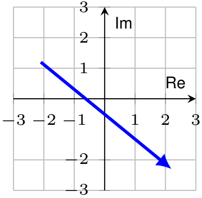


$$\sum_{k=0}^{n-1} X_k \cdot r_k + {\color{red} i}\; X_k \cdot i_k, \qquad r_k, i_k \in \mathbb{R}$$

$$\begin{array}{cccccccc} & X_0 \cdot r_0 & + & X_1 \cdot r_1 & + & X_2 \cdot r_2 & + & \dots \\ {\color{red} i} & X_0 \cdot i_0 & + & X_1 \cdot i_1 & + & X_2 \cdot i_2 & + & \dots \end{array}$$



+

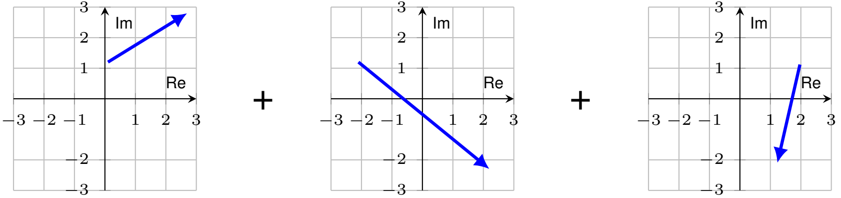


+

$$\sum_{k=0}^{n-1} X_k \cdot r_k + {\color{red}i}\; X_k \cdot i_k, \qquad r_k, i_k \in \mathbb{R}$$

$$\begin{array}{cccccccc} & \boxed{\begin{array}{c} X_0 \cdot r_0 \\ X_0 \cdot i_0 \end{array}} & \begin{array}{c} + \\ + \end{array} & \boxed{\begin{array}{c} X_1 \cdot r_1 \\ X_1 \cdot i_1 \end{array}} & \begin{array}{c} + \\ + \end{array} & \boxed{\begin{array}{c} X_2 \cdot r_2 \\ X_2 \cdot i_2 \end{array}} & \begin{array}{c} + \\ + \end{array} & \begin{array}{c} \dots \\ \dots \end{array} \end{array}$$

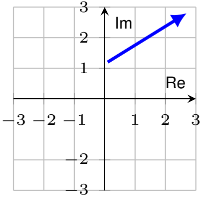
(with ${\color{red}i}$ multiplying the second row)



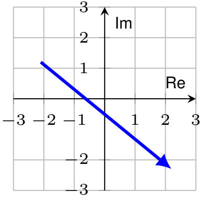
$$\sum_{k=0}^{n-1} X_k \cdot r_k + {\color{red} i}\; X_k \cdot i_k, \qquad r_k, i_k \in \mathbb{R}$$

$$\begin{array}{ccccccccc} & \boxed{\begin{array}{c} X_0 \cdot r_0 \\ X_0 \cdot i_0 \end{array}} & \begin{array}{c} + \\ + \end{array} & \boxed{\begin{array}{c} X_1 \cdot r_1 \\ X_1 \cdot i_1 \end{array}} & \begin{array}{c} + \\ + \end{array} & \boxed{\begin{array}{c} X_2 \cdot r_2 \\ X_2 \cdot i_2 \end{array}} & \begin{array}{c} + \\ + \end{array} & \begin{array}{c} \ldots \\ \ldots \end{array} \end{array}$$

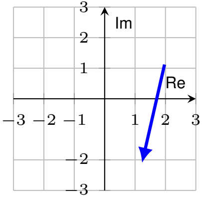
${\color{red} i}$ (second row)



+

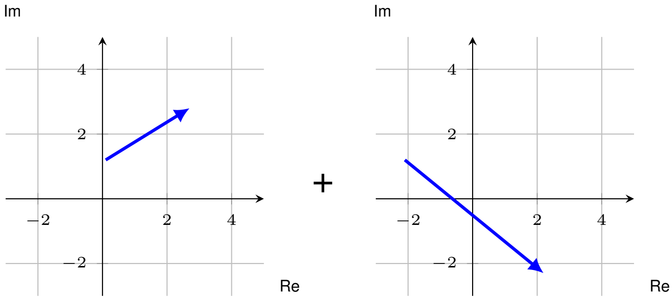


+

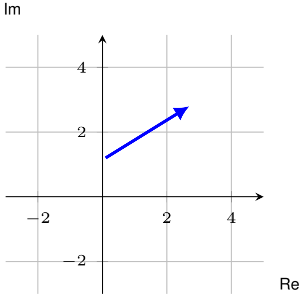


+ ...

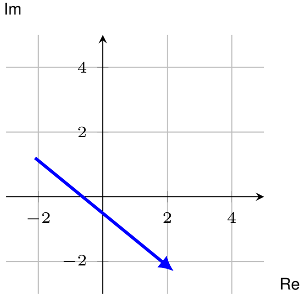
Sum of dependent intervals



Sum of dependent intervals

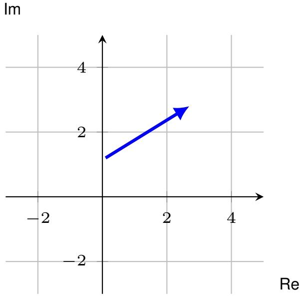


+

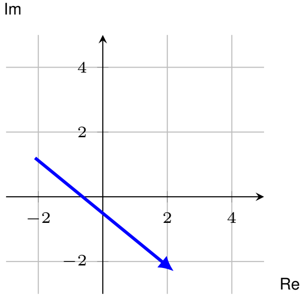


=

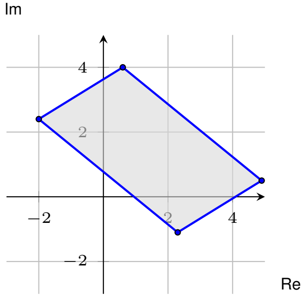
Sum of dependent intervals



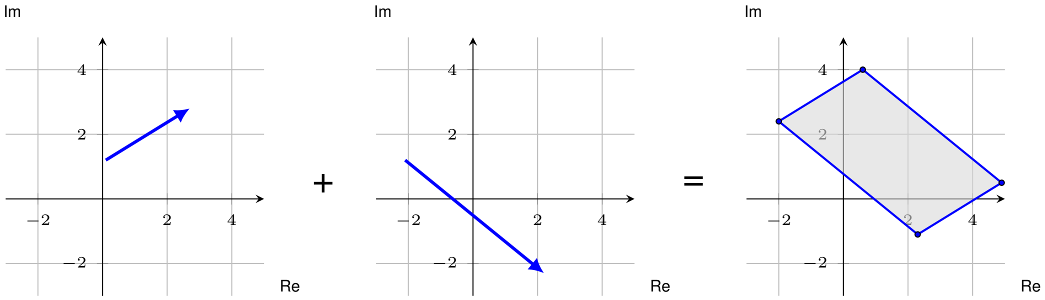
+



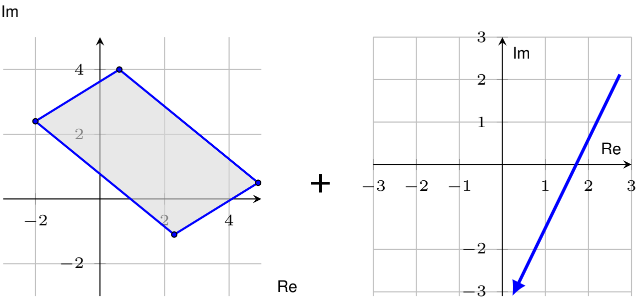
=

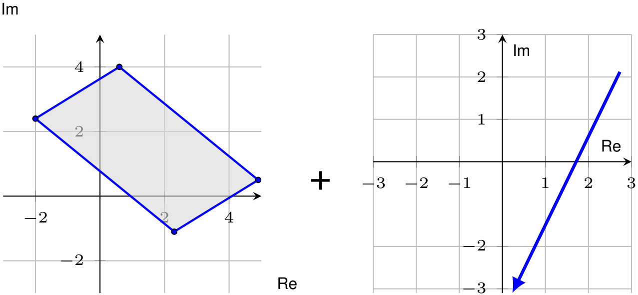


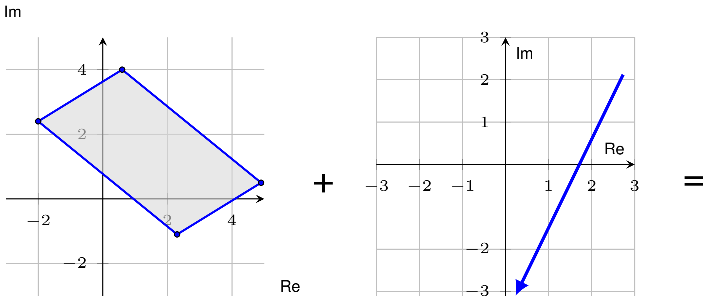
Sum of dependent intervals

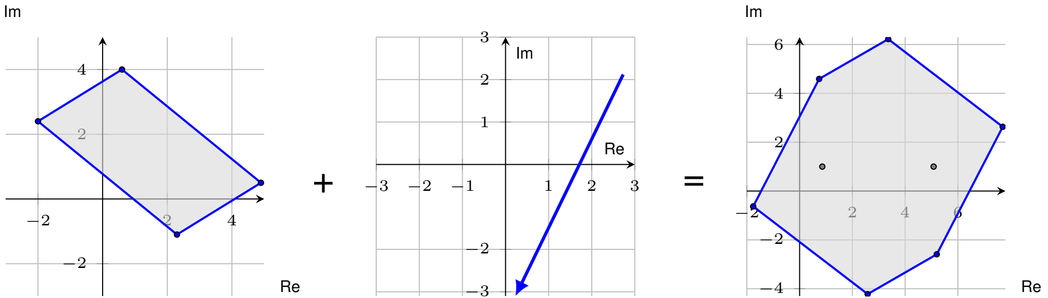


The sum is not closed w.r.t. the diagonals.


Im
4
2
−2
−2
2
4
Re
+
3
2
1
−2
−3
Im
Re
−3
−2
−1
1
2
3


Im
4
2
−2
−2
2
4
Re
+
3
2
1
−2
−3
Im
Re
−3
−2
−1
1
2
3


Im
4
2
−2
−2
2
4
Re
+
3
2
1
−2
−3
Im
Re
−3
−2
−1
1
2
3
=


Im
4
2
−2
2
4
−2
Re
+
3
2
1
Im
Re
−3
−2
−1
1
2
3
−2
−3
=
Im
6
4
2
−2
2
4
6
−2
−4
Re

Main theorems (Minkowski addition)

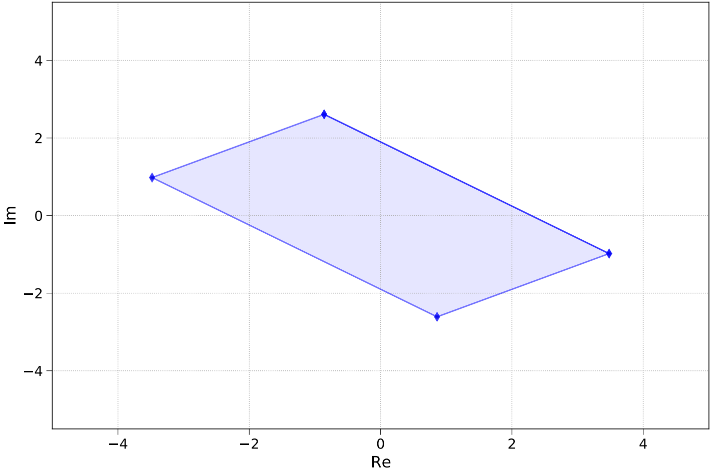
Theorem 3 (Linear transformation of convex sets)

Let $A \subseteq \mathbb{R}^n$ be a convex set and suppose that f is the linear map of $f : \mathbb{R}^n \to \mathbb{R}^m$. Then f maps the extreme points of A onto the extreme points of $f(A)$.

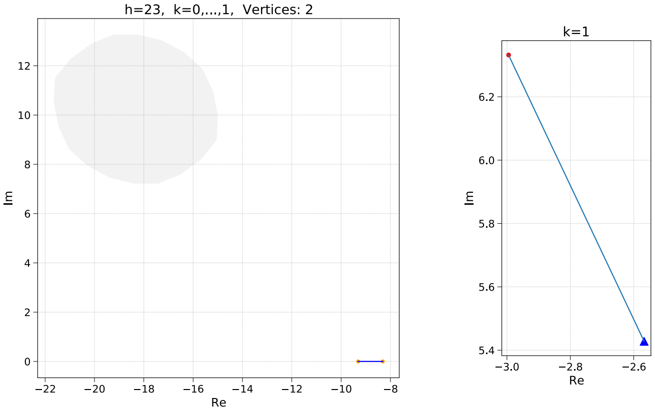
Theorem 4 (Sum of a convex set plus a segment)

The maximum distance between a convex polygon and a segment is attained at the endpoints of the segment.

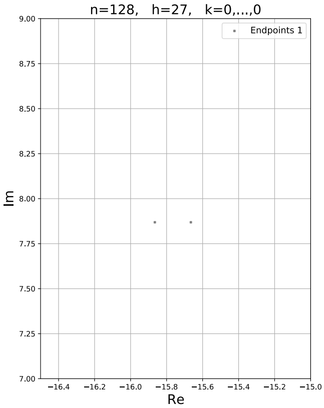
Proof (intuition)



Algorithm



Centered form and MC samples

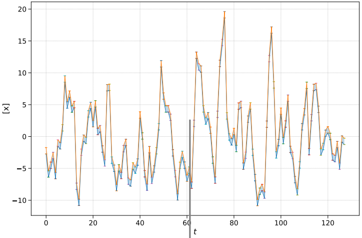


Computational cost and summary

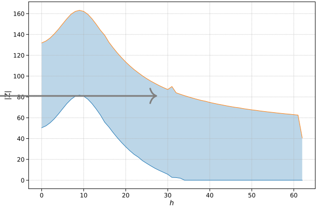
Brute Force $O(2^N)$

Combing algorithm $O(n^2 \log n)$

Interval extension $O(n)$



Interval Fourier transform

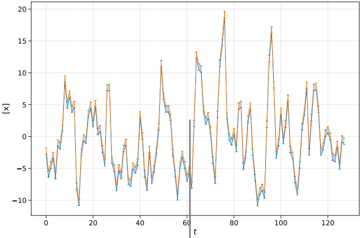


Computational cost and summary

Brute Force $O(2^N)$

Combing algorithm $O(n^2 \log n)$

Interval extension $O(n)$



Interval Fourier transform